1982-07

Interactive Markov chains and manpower modelling

Lehoczky, J. P.

Monterey, California. Naval Postgraduate School

NPS55-82-018

NAVAL POSTGRADUATE SCHOOL

Monterey, California

INTERACTIVE MARKOV CHAINS AND

MANPOWER MODELLING

by

J. P. Lehoczky

P. A. Jacobs

D. P. Gaver

July 1982

Prepared for:
Chief of Naval Research
Arlington, VA 22217

National Science Foundation
Washington, DC 20550

NAVAL POSTGRADUATE SCHOOL
Monterey, California

Rear Admiral J. J. Ekelund
Superintendent

David A. Schrady
Acting Provost

This work was supported in part by the National Science Foundation under Grant NSF-ENG-79-10825; and by the Office of Naval Research, Arlington, VA.

J. P. Lehoczky

J. P. Lehoczky, Professor
Carnegie-Mellon University

Peter A. Jacobs

P. A. Jacobs, Associate Professor
Department of Operations Research

Donald P. Gaver

D. P. Gaver, Professor
Department of Operations Research

Reviewed by:

K. T. Marshall

K. T. Marshall, Chairman
Department of Operations Research

Released by:

William M. Tolles

William M. Tolles
Dean of Research

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: NPS55-82-018	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): INTERACTIVE MARKOV CHAINS AND MANPOWER MODELLING	5. TYPE OF REPORT & PERIOD COVERED: Technical
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): J. P. Lehoczky, P. A. Jacobs, D. P. Gaver	8. CONTRACT OR GRANT NUMBER(s): NSF-ENG-79-10825
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Naval Postgraduate School, Monterey, CA 93940	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61153n; RR014-05-01 N0001481WR10001
11. CONTROLLING OFFICE NAME AND ADDRESS: Chief of Naval Research, Arlington, VA 22217; National Science Foundation, Washington, DC 20550	12. REPORT DATE: July 1982
	13. NUMBER OF PAGES: 21
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Discrete time manpower models; interactive Markov chains; Gaussian approximation; manpower control policies

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper introduces a class of discrete time manpower models based on interactive Markov chains. Individuals make transition decisions which are conditionally independent of each other given the total occupancy of the levels of the organizational hierarchy. This allows recruitment, promotion, and resignation to depend on the state of the system. A number of specific manpower models are introduced and analyzed using an asymptotic characterization of such systems. The last section considers the evaluation of recruitment and promotion policies when the organization seeks to reach some goal.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

INTERACTIVE MARKOV CHAINS

AND

MANPOWER MODELLING

by

J. P. Lehoczky[1]

P. A. Jacobs[2]

D. P. Gaver[3]

[1]Department of Statistics, Carnegie-Mellon University, Pittsburgh, PA 15213.
Research was supported in part from a Grant from the National Science Foundation NSF ECS 8101576.

[2]Department of Operations Research, Naval Postgraduate School, Monterey, CA 93940.
Research was supported in part by a Grant from the National Science Foundation ENG 7910825.

[3]Department of Operations Research, Naval Postgraduate School, Monterey, CA 93940.
Research supported in part by Office of Naval Research.

Abstract

This paper introduces a class of discrete time manpower models based on interactive Markov chains. Individuals make transition decisions which are conditionally independent of each other given the total occupancy of the levels of the organizational hierarchy. This allows recruitment, promotion, and resignation to depend on the state of the system. A number of specific manpower models are introduced and analyzed using an asymptotic characterization of such systems. The last section considers the evaluation of recruitment and promotion policies when the organization seeks to reach some goal.

1. Introduction

The development of mathematical models to aid in studying manpower planning strategies has occurred largely over the last twenty years. A fine summary of much of this work is given in Grinold and Marshall (1976). Some of the early work with stochastic models is contained in Bartholomew (1973). Generally, mathematical models have been deterministic and have focused on describing or controlling the age or grade structure in an organization. Stochastic models have generally not dealt with the impact of control strategies. Rather, they have attempted to develop models which capture the randomness of the recruitment process and the uncertainty associated with both promotion and resignation for the organization and the employee. Recent representative work of this type is by McClean (1976, 1978, 1980) or Mehlmann (1979). The McClean (1980) paper considers an organization having Poisson recruitment and departure process and semi-Markov transitions between grades. The models developed in these papers assume members of the organization act independently of each other and the organization; thus the models are often equivalent to networks of infinite server queues. As such these processes have product form equilibrium distributions with Poisson marginals -- see Kelly (1979) for example.

An alternate approach to stochastic modelling is carried out in Bartholomew (1977, 1979). These two papers introduce particular control strategies wherein the organization attempts to maintain a certain age or grade structure using recruitment or promotion control. Bartholomew studies the probability of success of certain strategies in a stochastic environment. This work represents an important step toward the development of stochastic models which can aid manpower planners in the development of successful policies.

Consider a large hierarchical organization. Individuals will be recruited into various levels or into only the lowest level as in the case of the military. The organization controls promotion and hiring. The employees can stay or leave, and thus with some exceptions control their own wastage. Nevertheless, the organization can surely influence the wastage process by policies affecting promotion, salary, and working conditions, etc. Generally, the organization wishes to maintain a certain age or grade structure or more generally wishes to achieve some goal. The goal might involve the total salary paid, the number at each level of the hierarchy, the quality of the workforce, or the number of minority employees among many other possibilities. Thus the organization attempts to meet its goal in the face of a stochastic wastage process. Deterministic models often lead to the conclusion that the goal can be reached exactly. Models which incorporate uncertainty show that this is rarely possible; see for example Bartholomew (1977, 1979).

In this paper, we develop Markovian models that partially capture the idea of dependent motions among employees. For example, if a particular level has too few occupants the organization will increase the promotion rate or recruitment rate (or both) into that level. Employees will generally decide to leave on the basis of individual factors but will also be influenced by perceived prospects, in part judged by the size of the various levels of the organization. If an employee's level or the higher levels are crowded, this may serve to discourage an employee and increase the probability of wastage. It is important to build these interactive effects into a stochastic model, since independent motions do not adequately represent most situations.

The models used in this paper generally belong to the class called "interactive Markov chains" introduced by Conlisk (1976, 1978) and extended by Gerchak and Brumelle (1980) and Lehoczky (1980). These models also generally correspond to Markov population processes as developed by Kingman (1969) and discussed extensively by Kelly (1979), but this work focuses on the reversible or quasi-reversible cases and deals solely with equilibrium behavior. We will be interested in studying cases which do not necessarily satisfy the reversibility conditions and which exhibit transient behavior.

Section 2 provides a brief description of discrete time interactive Markov chains. Section 3 gives a description of an asymptotic analysis. Section 4 presents several specific manpower models. Section 5 introduces some control policy aspects.

2. Discrete Time Interactive Markov Chains

We consider a large population of N individuals. Each individual may be in one of $K+1$ distinct states $\{0,1,\ldots, K\}$. In manpower applications, states $1,\ldots, K$ represent the levels of a hierarchy, while state 0 represents individuals outside the organization. As such, this would be a closed system. Let $X_{Ni}(t)$ = number of individuals in state i at time t, $t = 0,1,\ldots$, $0 \leq i \leq K$ and define $\underset{\sim}{X}_N(t) = (X_{N0}(t), \ldots, X_{NK}(t))$. The $\underset{\sim}{X}_N$ process is K dimensional, since $\sum_{i=0}^{K} X_{Ni}(t) = N$, $\forall t$. We wish to consider models for which the $\{\underset{\sim}{X}_N(t), t = 0,1,\ldots\}$ process is Markovian. We assume each of the N individuals make independent transition decisions from time t to $t+1$, but these decisions depend on $\underset{\sim}{X}_N(t)$. This allows individuals within the organization to base departure decisions on the current system configuration; however, these decisions are made independently. This allows individuals to respond to crowding or emptiness at any level. Similarly, the recruitment can also depend on the occupancy of the various levels. If $X_{N_i}(t) = \ell$, then each of these ℓ individuals will independently make a transition and will move to state j with probability $p_{ij}(\underset{\sim}{X}_N(t)/N)$. The functions $\{p_{ij}(\underset{\sim}{x}) , 0 \leq i,j \leq K\}$ are defined on the unit simplex $\{\underset{\sim}{x} \mid x_i \geq 0 , \sum_{i=0}^{K} x_i = 1\}$ and are assumed to be continuously differentiable in x_i , $0 \leq i \leq K$. It is convenient to think of each individual as having the trajectory of a discrete time Markov chain with transition matrix $\underset{\sim}{P}(\underset{\sim}{X}_N(t)/N)$ conditional on $\underset{\sim}{X}_N(t)$. Clearly we assume $p_{ij} \geq 0$ and $\sum_{j=0}^{K} p_{ij} = 1$ for each value of $\underset{\sim}{X}_N(t)$.

The class of models utilized in this paper and the analysis presented can be easily generalized in several useful ways. First, it is possible to allow an individual's transition decision at time t to depend on the system configuration at times $t-\tau$, $t-\tau+1, \ldots,$ $t-1$, and t rather than just $\underset{\sim}{X}_N(t)$. One simply adds $\underset{\sim}{X}_N(t-\tau), \ldots,$ $\underset{\sim}{X}_N(t-1)$ as supplementary variables to induce a Markov process. Second, one can remove the condition that the system be closed, that is, the number of individuals in the $K + 1$ states need not be constant. This allows for compound Poisson input processes, and the resulting process will no longer be confined to a unit simplex. An example of this type is given by Model 2 of section 4.

The interactive Markov chains used in this paper are all discrete time processes. A similar analysis can be developed in continuous time and is given in Lehoczky (1980). The discrete time formulation is more appropriate in many manpower contexts.

3. Asymptotic Analysis

The interactive Markov chain models defined in section 2 provide a rich class of models, but the models may be difficult to analyze. Generally there are two approaches that can be carried out. First, certain interactive Markov chains are reversible. The equilibrium analysis of this class has been thoroughly studied by Kingman (1969) and Kelly (1979). Typically, this class generates a joint Poisson equilibrium distribution. Our manpower models occasionally fall into this category, thus yielding a simple analysis. Unfortunately, our models will frequently not be reversible.

A second method can be employed for large populations sizes N. This involves finding an asymptotic approximation for $\underset{\sim}{X}_N(t)$ as $N \to \infty$. Frequently, this will be of the form $\underset{\sim}{X}_N(t) = N\underset{\sim}{m}(t) + N^{\frac{1}{2}}Z(t) + o_p(N^{\frac{1}{2}})$ where $m(t)$ is a deterministic vector $(m_0(t), \ldots, m_K(t))$ of discrete time functions satisfying $0 \leq m_i(t) \leq 1$ and $\sum_{i=0}^{K} m_i(t) = 1$. The process $\{\underset{\sim}{Z}(t), t=0,1,\dots\}$ will generally be a $K + 1$ dimensional diffusion process with singular Gaussian distribution for $\underset{\sim}{Z}(t)$. Further, $o_p(N^{\frac{1}{2}})/N^{\frac{1}{2}} \overset{P}{\to} 0$ as $N \to \infty$.

The asymptotic expansion of $\underset{\sim}{X}_N(t)$ is useful both for an equilibrium analysis and for insight into the transient behavior of the system. Conlisk (1976) studied the behavior of this class of stochastic systems through $\underset{\sim}{m}(t)$ alone. The function $\underset{\sim}{m}(t)$ is a deterministic approximation to $\underset{\sim}{X}_N(t)/N$ for large N. Conlisk (1976) assumes, and Gerchak and Brumelle (1980) prove that $\underset{\sim}{X}_N(t)/N \overset{P}{\to} \underset{\sim}{m}(t)$ where $\underset{\sim}{m}(t)$ satisfies

$$\underset{\sim}{m}(t + 1) = \underset{\sim}{m}(t)\ \underset{\sim}{P}(\underset{\sim}{m}(t)) \tag{3.1}$$

with $\underset{\sim}{m}(0) = \underset{\sim}{y}$ as initial condition and $\underset{\sim}{P}(\underset{\sim}{m}(t)) = (p_{ij}(\underset{\sim}{m}(t)))$. Equation (3.1) is typically nonlinear, so a wide variety of system behaviors can be exhibited. Furthermore, $\underset{\sim}{m}(t)$ may be different from $E(\underset{\sim}{X}_N(t))$, thus the expansion of $\underset{\sim}{X}_N(t)$ is made about the deterministic approximation, $\underset{\sim}{m}(t)$, rather than the $E(\underset{\sim}{X}_N(t))$. If the system is closed, so $\underset{\sim}{m}$ is restricted to the unit simplex, the Brouwer fixed point theorem guarantees (3.1) has at least one, and possibly several, fixed points. These fixed points may be either stable or unstable, and Conlisk carefully analyzes the various possibilities for a special class of p_{ij} functions. For open systems, in which $\underset{\sim}{m}$ is no longer confined to the unit simplex, there may be no fixed points and possibly no equilibrium distribution.

The assumption that individuals make independent transition decisions conditional on the system configuration induces both a strong law of large numbers and a central limit approximation. In our model, we assume that, given $\underset{\sim}{X}_N(t)$, the $X_{Ni}(t)$ individuals in group i move to j according to a binomial $(X_{Ni}(t), p_{ij}(\underset{\sim}{X}_N(t)/N))$ distribution. If N is large, this will be approximately Gaussian by the central limit theorem. The occupancy of group j at time $t+1$ is, conditional on $\underset{\sim}{X}_N(t)$, the superposition of $K+1$ independent binomials which will again be approximately Gaussian. Furthermore, conditional on $\underset{\sim}{X}_N(t)$, $\underset{\sim}{X}_N(t+1)$ will be the sum of $K+1$ independent multinomial vectors. The conditional distribution of $\underset{\sim}{X}_N(t+1)$ can be approximated by a singular $K+1$ dimensional normal. The proof of these results is given in Lehoczky (1980).

The above facts encourage the manpower planner to study manpower problems using tools of multivariate Gaussian time series analysis. Application of methods of spectral analysis, including filtering and prediction techniques, may well prove effective and informative in practice.

Suppose we adopt an initial condition for the process, $\underset{\sim}{X}_N(0)$, given by a multinomial distribution with parameters N and $\underset{\sim}{y}$, $0 \leq y_i \leq 1$., $\sum_{i=0}^{K} y_i = 1$. It follows that $\underset{\sim}{X}_N(0)$ will have a covariance matrix given by $N\underset{\sim}{\Sigma}_0$ where $\underset{\sim}{\Sigma}_0 = (\sigma_{ij}(0))$ and $\sigma_{ii}(0) = y_i(1-y_i)$, $\sigma_{ij}(0) = -y_i y_j$, $i \neq j$. The vector sum $(\underset{\sim}{X}_N(0) - N\underset{\sim}{y})/N^{\frac{1}{2}}$ will be approximately Gaussian with mean $\underset{\sim}{0}$ and covariance matrix $\underset{\sim}{\Sigma}_0$. One might simply initialize the interactive Markov chain with a Gaussian distribution having mean $N\underset{\sim}{y}$ and covariance $N\underset{\sim}{\Sigma}_0$ where $\underset{\sim}{\Sigma}_0$ is any singular covariance matrix.

If the initial distribution of $\underset{\sim}{X}_N(0)$ is deterministic or Gaussian, and the conditional distributions of $\underset{\sim}{X}_N(t)$ are Gaussian, then the marginal distributions of $\underset{\sim}{X}_N(t)$ will also be Gaussian. One can find the sequence of means for the $\{\underset{\sim}{X}_N(t)\}$ process from (3.1) with $\underset{\sim}{m}(0) = \underset{\sim}{y}$. The covariance sequence satisfies

$$\underset{\sim}{\Sigma}_{t+1} = \underset{\sim}{B}(\underset{\sim}{m}(t)) + \underset{\sim}{G}^T(\underset{\sim}{m}(t))\ \underset{\sim}{\Sigma}_t\ \underset{\sim}{G}(\underset{\sim}{m}(t)) \tag{3.2}$$

where the elements of $\underset{\sim}{B}$ are

$$b_{ii}(\underset{\sim}{y}) = \sum_{k=0}^{K} y_k p_{ki}(\underset{\sim}{y})(1 - p_{ki}(\underset{\sim}{y})) \tag{3.3}$$

$$b_{ij}(\underset{\sim}{y}) = -\sum_{k=0}^{K} y_k p_{ki}(\underset{\sim}{y}) p_{kj}(\underset{\sim}{y}), \quad i \neq j \tag{3.4}$$

and

$$\underset{\sim}{G}(\underset{\sim}{y}) = \underset{\sim}{A}(\underset{\sim}{y}) + \underset{\sim}{P}(\underset{\sim}{y}) \tag{3.5}$$

with

$$a_{ij}(\underset{\sim}{y}) = \sum_{k=0}^{K} y_k \left(\frac{\partial}{\partial y_i} P_{kj}(\underset{\sim}{y})\right). \tag{3.6}$$

The derivations are given by Lehoczky (1980).

The Gaussian characterization of $\underset{\sim}{X}_N(t)$ allows one to examine the transient behavior. If at any time t, $\underset{\sim}{X}_N(t) = N\underset{\sim}{x}$, then one can begin with initial condition $\underset{\sim}{X}_N(t) = N\underset{\sim}{x}$ and $\underset{\sim}{\Sigma}_t = 0$. Equations (3.1) - (3.6) can be used to calculate the mean and covariance matrix at any future time. In addition, the asymptotic characterization allows one to construct a likelihood function based on the normal distribution to study the parameters of the wastage distribution.

4. Manpower Models

In this section, we introduce several specific manpower planning models to illustrate the use of interactive Markov chains.

Model 1:

In this case we study a closed system with independent motions. We assume a hierarchical organization with recruitment into only the lowest level, number 1. The levels are $1,\ldots, K$ with 0 representing the outside. We assume there is no demotion. In such a case, one might assume all transition probabilities are independent of the system state and

$$\begin{aligned}
&p_{01} = \lambda \ , \quad p_{00} = 1 - \lambda \\
&p_{ii+1} = p_i \ , \quad p_{i0} = w_i \ , \quad p_{ii} = 1 - p_i - w_i \quad \text{with } p_i + w_i \le 1 \ , \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \le i \le K - 1 \\
&p_{K0} = w_K \ , \quad p_{KK} = 1 - w_K \ .
\end{aligned} \tag{4.1}$$

This probability assignment has a simple interpretation. Individuals outside the organization have a probability λ of joining, while individuals in level i have a probability p_i of promotion and w_i of resignation. The deterministic process $\underset{\sim}{m}(t)$ of (3.1) is given by

$$\begin{aligned}
m_1(t+1) &= \lambda m_0(t) - (p_1 + w_1)m_1(t) \\
m_i(t+1) &= p_{i-1}m_{i-1}(t) - (p_i + w_i)m_i(t) \ , \quad 2 \le i \le K - 1 \\
m_K(t+1) &= p_{K-1}m_{K-1}(t) - w_K m_K(t) \\
m_0(t+1) &= 1 - \sum_{i=1}^{K} m_i(t+1) \ .
\end{aligned} \tag{4.2}$$

This system is identical to a closed Markovian network of infinite server queues. The equilibrium mean term is given by

$$\overline{m}_i = \prod_{j=1}^{i} \alpha_j \Big/ \left(1 + \sum_{i=1}^{K} \prod_{j=1}^{i} \alpha_j\right) \ , \quad 0 \le i \le K \tag{4.3}$$

where $\alpha_1 = \lambda/(1 + w_1 + p_1)$, $\alpha_i = p_{i-1}/(1 + p_i + w_i)$, $2 \leq i \leq K - 1$, $\alpha_K = p_{K-1}/w_K$, and the numerator of (4.1) is taken to be 1 when $i = 0$.

The transition probabilities are not state dependent, so the matrix $\underset{\sim}{A}$ of (3.6) is identically $\underset{\sim}{0}$. The $\underset{\sim}{G}$ matrix of (3.5) is given by $\underset{\sim}{P}$ alone. The matrix $\underset{\sim}{B}$ is easily constructed, thus (3.2) can be used to find the covariance matrix at any time t. The covariance matrix associated with the equilibrium distribution, Σ_∞, is easily determined. For $K = 2$, it is the covariance matrix associated with a multinomial distribution having parameter N and probability vector

$(w_1 + p_1, \lambda, \lambda p_1/w_2)/(w_1 + p_1 + \lambda + \lambda p_1/w_2)$.

This analysis provides a useful characterization of the system for large N. As an alternative one could simply treat this as a Gordon-Newell network having Poisson product form equilibrium distribution restricted to the unit simplex.

Model 2:

A simple variation on Model 1 occurs when one allows independent motions with an open system. In this case, the input from outside the organization in any time interval is assumed to be an independent Poisson random variable with parameter $N\theta$. The transition probabilities are given by (4.1) for $i = 1, \ldots, K$. This then corresponds to a network of infinite server queues in discrete time. The model becomes a Jackson network and hence the equilibrium distribution will be of product form with Poisson marginals. Although this open model is somewhat different from the closed model 1, the asymptotic results of section 3 remain the same.

The asymptotic analysis would generate equations similar to (4.2) for $\{m(t), t = 0,1,\ldots\}$ given by

$$\begin{aligned} m_1(t+1) &= \theta + (1 - p_1 - w_1)m_1(t) \\ m_i(t+1) &= p_{i-1}m_{i-1}(t) + (1 - p_i - w_i)m_i(t), \; 2 \le i \le K \end{aligned} \qquad (4.4)$$

with $p_K = 0$.

The process is no longer on the unit simplex. In equilibrium, the occupancy of level 1 has Poisson $(N\theta/(p_1 + w_1))$ distribution, while level 2 occupancy has an independent Poisson $(Np_1\theta/w_2)$ distribution for the case $K = 2$. Asymptotically, the equilibrium distribution is approximated by

$$N(\theta/(p_1 + w_1) \; , \; p_1\theta/w_2) + N^{\frac{1}{2}} \mathcal{N}\left((0,0) \; , \begin{pmatrix} \theta/(p_1 + w_1) & 0 \\ 0 & p_1\theta/w_2 \end{pmatrix}\right) \qquad (4.5)$$

where $\mathcal{N}$ represents a bivariate normal distribution. Similar formulas apply when $K > 2$.

Both Models 1 and 2 are seriously flawed as models for manpower planning purposes. Generally, organizations will modify recruitment and promotion decisions depending on the current state of the system. If level 1 is small, recruitment (represented by P_{01} or θ) will be increased. If level 1 is large, recruitment will be slowed and promotion to 2 increased. Similarly, when both levels are full, individuals tend to become discouraged about their careers. The wastage parameters will then increase. Models which incorporate some of these effects will be studied next.

Model 3:

In this model, we assume that there are two levels in the hierarchy and that the organization has numerical goals G_1 and G_2 for the sizes of each level. That is, the organization wishes to reach and maintain G_i individuals in each level i. We will assume that employees do not base their departure decisions on the state of the system, instead their wastage probabilities are given by w_1 and w_2 as in Models 1 and 2. If one adopted a deterministic model with w_1 and w_2 known, the organization would promote $G_2 - m_2(t)(1 - w_2)N$ from 1 to 2 provided this were feasible, i.e. $0 \le G_2 - m_2(t)(1 - w_2)N \le m_1(t)(1 - w_1)N$. Similarly, the organization would recruit $G_1 - m_1(t)(1 - w_1)N + G_2 - m_2(t)(1 - w_2)N$ into level 1 provided this quantity were positive and less than $Nm_0(t)$. We can create a stochastic model which captures this policy using interactive Markov chains. Specifically,

$$\begin{aligned} p_{01}(\underset{\sim}{m}(t)) &= \frac{g_1 + g_2 - m_1(t)(1 - w_1) - m_2(t)(1 - w_2)}{m_0(t)} \\ p_{12}(\underset{\sim}{m}(t)) &= \frac{g_2 - m_2(t)(1 - w_2)}{m_1(t)} \end{aligned} \qquad (4.6)$$

$$p_{10}(\underset{\sim}{m}(t)) = w_1 \quad , \quad p_{20}(\underset{\sim}{m}(t)) = w_2$$

with all other p_{ij}'s being 0. Furthermore $g_i = G_i/N$.

These p_{01} and p_{12} transition probabilities may require modification to ensure that $\underset{\sim}{P}(\underset{\sim}{m}(t))$ is a bona fide transition matrix, that is $p_{ij}(\underset{\sim}{m}(t)) \ge 0$ and $\sum_j p_{ij}(\underset{\sim}{m}(t)) = 1$. These modifications may cause p_{ij} to become nondifferentiable. We deal only with cases where p_{ij} is evaluated at points of differentiability.

If we begin with $\underset{\sim}{m}(0)$ such that $m_2(0)(1-w_2) < g_2 <$ $m_2(0)(1-w_2) + m_1(0)(1-w_1) < g_1 + g_2$ and $m_0(0) \geq g_1 + g_2 - (m_1(0)(1-w_1) + m_2(0)(1-w_2))$, then $\underset{\sim}{m}(1) = \underset{\sim}{m}(t) = (g_1, g_2)$ for all t. The organization is immediately able to move the deterministic approximation to its goal. The covariance structure of the associated Gaussian process will, however, be quite complicated although straightforward to compute using (3.2) - (3.6). For suitable $\underset{\sim}{m}(0)$, $\underset{\sim}{m}(t)$ will be (g_1, g_2) for $t \geq 1$. This means that the covariance matrix $\underset{\sim}{\Sigma}_t$ will be evaluated at (g_1, g_2) as well, thus simplifying the calculations.

<u>Model 4</u>:

In model 4, we consider an open system with two levels where the number recruited into level 1 at time t has a Poisson distribution with mean $N[w_1 m_1(t) + w_2 m_2(t)]$. We assume $p_{20}(t) = w_2$ as in Model 3. We assume the wastage probability from rank 1 at time t depends on the state $\underset{\sim}{m}(t-1)$ in the following manner

$$p_{10}(\underset{\sim}{m}(t-1)) = w_1 + \frac{w_2 g_2}{g_1} - \frac{w_2 m_2(t-1)}{m_1(t-1)} - \frac{g_2 - m_2(t-1)}{m_1(t-1)} \qquad (4.7)$$

where p_{10} is taken to be 0 if (4.7) gives a negative value, and 1 if (4.7) exceeds 1. The promotion probability from rank 1 to 2 is

$$p_{12}(\underset{\sim}{m}(t)) = \frac{g_2 - (1-w_2)m_2(t)}{m_1(t)} \qquad (4.8)$$

truncated to the interval $[0, 1 - p_{10}(\underset{\sim}{m}(t))]$. The promotion probability from rank 1 to 2 is $w_2 g_2/g_1$ if the system is at its goal and

$\frac{(g_2 - m_2(t-1))}{m_1(t-1)} + w_2 m_2(t-1)/m_1(t-1)$ is the actual promotion probability at time $t-1$. Hence $p_{10}(t)$ tends to be larger (smaller) than w_1 if the promotion probability at time $t-1$ is smaller (larger) than the "usual" $w_2 g_2/g_1$. The organization is unaware of the true wastage probability p_{10} and uses w_1 as an estimate for the wastage probability in determining the mean number recruited in the t^{th} time interval which is $N[w_1 m_1(t) + w_2 m_2(t)]$.

If $m_i(0) = g_i$ for $i = 1,2$, then $m_i(t) = g_i$ for $i = 1,2$ and $t \geq 1$ as before. The Gaussian terms $(Z_1(t), Z_2(t))$ satisfy the system

$$
\begin{aligned}
Z_1(t+1) &= W_{01}(t) + \sqrt{g_1}\, W_{11}(t) - a_1 Z_1(t-1) - (1-w_2) Z_2(t-1) \\
Z_2(t+1) &= \sqrt{g_1}\, W_{12}(t) + \sqrt{g_2}\, W_{22}(t)
\end{aligned}
\tag{4.9}
$$

where $a_1 = w_2 g_2/g_1$, $b_1 = 1 - a_1 - w_1$ and $W_{01}(t), (W_{11}(t), W_{12}(t)), W_{22}(t)$ for $t \geq 0$ are independent normal random variables with mean 0, and

$$\mathrm{Var}(W_{01}(t)) = w_1 g_1 + w_2 g_2$$
$$\mathrm{Var}(W_{22}(t)) = (1-w_2) w_2$$
$$\mathrm{Var}(W_{11}(t)) = b_1(1-b_1)$$
$$\mathrm{Var}(W_{12}(t)) = a_1(1-a_1)$$
$$\mathrm{Cov}(W_{11}(t), W_{12}(t)) = -b_1 a_1$$

Some second moments of $\underset{\sim}{Z}(t)$ for $t = 1,2,\ldots$ are

$$\mathrm{Var}(Z_2(t)) = g_1 a_1(1-a_1) + g_2 w_2(1-w_2)$$
$$\mathrm{Cov}(Z_1(t), Z_2(t)) = -g_1 a_1 b_1$$
$$\mathrm{Cov}(Z_2(t+s), Z_2(t)) = \mathrm{Cov}(Z_2(t+s), Z_1(t)) = 0$$

for $s \neq 0$.

By letting $t \to +\infty$, we compute the following asymptotic second moments

$$\lim_{t\to\infty} \operatorname{Var}(Z_1(t)) = (1-a_1^2)\Big\{w_1g_1 + w_2g_2 + g_1a_1(1-a_1) + (1-w_2)^2 \operatorname{Var}(Z_2(t)) - 2a_1(1-w_2)g_1b_1a_1\Big\} ,$$

$$\lim_{t\to\infty} \operatorname{Cov}(Z_1(t+2n), Z_1(t)) = -a_1^{n-1}\Big\{a_1 \lim_{t\to\infty} \operatorname{Var}(Z_1(t)) - g_1b_1a_1(1-w_2)\Big\} \qquad (4.12)$$

for $n = 1,2,\ldots$, and

$$\lim_{t\to\infty} \operatorname{Cov}(Z_1(t+2n+1), Z_1(t)) = 0 \quad \text{for } n \geq 0 .$$

If the above model were altered so that the number recruited in the t^{th} time interval has a Poisson distribution with mean $N(g_1p_{10}(t) + w_2g_2)$, (that is the organization knows $p_{10}(t)$), then under the same initial conditions the Gaussian approximation terms $\underset{\sim}{Z}(t)$, $t = 1,2\ldots$ are iid random variables with

$$\begin{aligned} \operatorname{Var}(Z_1(t)) &= w_1g_1 + w_2g_2 + g_1b_1(1-b_1) , \\ \operatorname{Cov}(Z_1(t), Z_2(t)) &= -g_1b_1a_1 , \qquad (4.13) \\ \operatorname{Var}(Z_2(t)) &= w_2g_2(2-a_1-w_2) . \end{aligned}$$

It follows that the organization's use of w_1 as an estimate of the unknown $p_{10}(t)$ affects only $Z_1(t)$, $t = 1,2,\ldots$. It changes the variance and introduces a cyclic behavior in $\operatorname{Cov}(Y_1(t+s), Y_1(t))$. Since the organization knows the constant attrition probability w_2 and can adjust $p_{12}(t)$, $Z_2(t)$, $t = 1,2,\ldots$ is unaffected.

5. Manpower Planning

In the deterministic approach to manpower planning, the wastage, recruitment, and promotions are often assumed to be known or controlled with certainty. In general, however, these flows are stochastic and thus cannot be known with certainty. Moreover, the organization generally does not even know the wastage probabilities, although they can be estimated from previous wastage data.

In the previous sections, we have illustrated how one might formulate particular manpower policies using interactive Markov chains. The models are analytically tractable if a normal approximation is used. In this section, we describe how one might use this formulation to evaluate the cost associated with any particular policy and thus find a policy to minimize the cost. These considerations necessitate that we first develop a decision-theoretic structure.

In general, a hierarchical organization will have several levels, and the staffing at each level for the n^{th} period can be represented by $\underset{\sim}{X}_n$. We assume that the organization has a goal, an ideal staffing level, for each level in each time period. Specifically, $\underset{\sim}{G}_n$ represents the goal vector for the n^{th} period. Suppose a particular policy results in a level $\underset{\sim}{x}_n$ for the n^{th} period. There will be some loss associated with this situation (a goal of $\underset{\sim}{G}_n$ and an actual level $\underset{\sim}{x}_n$). If a particular level is overstaffed $(x_{ni} > G_{ni})$, then the organization will be paying too much in salary. If a level is understaffed, $(x_{ni} < G_{ni})$, then certain functions may be inadequately performed. This may require outside services to be sought, and may cause a drop in productivity. We assume that i is possible to measure these losses numerically and postulate the existence of a loss function which does so. Specifically, let $L_n(\underset{\sim}{x}, \underset{\sim}{G})$ denote the loss associated with actual level $\underset{\sim}{x}$ and goal $\underset{\sim}{G}$ for the n^{th} time period. Let $L_n(\underset{\sim}{G}, \underset{\sim}{G}) = 0$, and assume $L_n(\underset{\sim}{x}, \underset{\sim}{G})$ to be convex in $\underset{\sim}{x}$ for fixed $\underset{\sim}{G}$. In a deterministic formulation, it is often possible to choose $\underset{\sim}{x}_n = \underset{\sim}{G}$. In a stochastic formulation, this no longer holds. One is interested in finding a policy which minimizes some objective function such as $\sum_{i=0}^{T} \beta^i E(L_i(\underset{\sim}{X}_i, \underset{\sim}{G}_i))$ where $0<\beta\leq 1$ is some discount factor. The $\{\underset{\sim}{X}_n\}$ process will be modelled by an interactive Markov chain as discussed in section 2. This places manpower planning into the general framework of Markov decision theory (see Ross (1970) for a general introduction), except the stochastic process is more complicated, being an interactive Markov chain. This formulation ignores the cost of implementation of a policy, but this can be easily added.

To illustrate these ideas, we consider a single period single compartment model. Suppose we have a current level x_0 in the compartment and a probability p that individuals will resign during the period. We must recruit some number of individuals to reach a goal G. We call this number r, and the manpower level in the next time period has approximately a normal distribution with mean $(1 - p)x_0 + r$ and variance $p(1 - p)x_0$. This assumes no variance associated with hiring replacements. Suppose the goal is given by G and $L(x, G) = (x - G)^2$. We thus seek $r \geq 0$ to minimize $E(X_1 - G)^2) = (px_0 + r - G)^2 + p(1 - p)x_0$. Clearly, the optimal $\hat{r} = g - (1 - p)x_0$ provided this is nonnegative or $\hat{r} = 0$. This is in accord with a deterministic formulation of the control problem as well, and this solution will be the same for any loss function symmetric about G.

The optimal choice of r may change if the loss function is no longer symmetric. For example, two reasonable loss functions might be

$$L_1(x, G) = \begin{cases} c_1(G - x) & \text{if } x \leq G \\ c_2(x - G) & \text{if } x \geq G , \end{cases} \tag{5.1}$$

or

$$L_2(x, G) = \begin{cases} h_1(G - x) & \text{if } G - x > l_1 \\ h_2(x - G) & \text{if } x - G > l_2 \\ 0 & \text{if } l_1 + G \leq x \leq l_2 + G . \end{cases} \tag{5.2}$$

In the first case, (5.1), the loss is proportional to the distance between the goal and the level reached, however, the constants of proportionality depend on whether too high or too low a level is achieved. The optimal r in such a case is given by $\hat{r} = G - (1 - p)x_0 + p(1 - p)x_0\Phi^{-1}(c_1/(c_1 + c_2))$, where Φ is the standard normal c.d.f. provided this quantity is nonnegative.

The second loss function given by (5.2) indicates that the goal G need not be exactly attained, rather there is an interval about G within which there is no loss. The loss for a result outside this interval will depend on the magnitude and the sign of the error. The optimum r will, of course, depend on the choice of l_1, l_2, h_1, and h_2 .

The first example showed that the choice of loss function can lead to changes in the choice of optimal r. The choice of r will also depend on whether one considers a single or multi-period problem formulation. Suppose we consider the same model, but seek to minimize over a two period time frame with discount factor β. Specifically, let x_0 be the original level at time 0 and r_i be the number of recruits for period $i = 1, 2$.

The goal is given by G in both time periods, and we seek r_1 and r_2 to minimize

$$E((X_1 - G)^2)^2 + \beta E((X_2 - G)^2) . \tag{5.3}$$

For simplification, we assume that the r's may be negative. In general, if an optimal r is negative we truncate it to be 0, but in multiperiod problems, this causes problems of tractability. We have

$$X_1 \sim N((1 - p)x_0 + r_1 , \; p(1 - p)x_0) . \tag{5.4}$$

$$X_2 | X_1 \sim N((1 - p)x_1 + r_2 , \; p(1 - p)x_1) .$$

Using (5.4), we calculate (5.3) using $\hat{r}_2(x_1) = G - px_1$ to be

$$(px_0 + r_1 - G)^2 + p(1 - p)x_0 + \beta p(1 - p)E(X_1) \tag{5.5}$$

where $E(X_1) = px_0 + r_1$.

The optimal choice of r_1 is given by $G - (1 - p)x_0 - \frac{1}{2}\beta p(1 - p)$, rather than $G - (1 - p)x_0$. The modification is slight since $\frac{1}{2}\beta p(1 - p)$ is small, but this illustrates that policies other than the usual deterministic policy will be called for using this decision theoretic framework.

The two examples presented were to illustrate that in a stochastic environment the deterministic policy of making the mean equal the goal is not optimal even in the simplest setting. When multiperiod problems are studied with multilevel organizations, far different policies may be called for. The important point is that interactive Markov chains provide a modelling tool to describe manpower phenomenon, and an asymptotic theory exists to allow for a tractable analysis of such problems.

REFERENCES

Bartholomew, D. J. Stochastic Models for Social Processes. Second Edition. J. Wiley and Sons, New York, 1973.

____________. Maintaining a grade or age structure in a stochastic environment. Adv. Appl. Prob. 9 (1977) 1-17.

____________. The control of a grade structure in a stochastic environment using promotion control. Adv. Appl. Prob. 11 (1979) 603-615.

Conlisk, J. Interactive Markov chains. J. Math. Sociology 4 (1976) 157-185.

____________. A stability theorem for an interactive Markov chain. J. Math. Sociology 6 (1978) 163-168.

Gerchak, Y. and Brumelle, S. A stochastic model allowing interaction among individuals and its behavior for large populations. J. Math. Sociology 7 (1980) 73-90.

Grinold, R. C. and Marshall, K. T. Manpower Planning Models. Elsevier/ North-Holland, New York, 1976.

Kelly, F. P. Reversibility and Stochastic Networks. J. Wiley and Sons, New York, 1979

Kingman, J. F. C. Markov population processes. J. Appl. Prob. 6 (1969) 1-18.

Lehoczky, J. P. Approximations for interactive Markov chains in discrete and continuous time. J. Math Sociology 7 (1980) 139-157.

McClean, S. I. A continuous-time population model with Poisson recruitment. J. Appl. Prob. 13 (1976) 348-354.

McClean, S. I. Continuous-time stochastic models of a multigrade population. J. Appl. Prob. 15 (1978) 26-32.

McClean, S. I. A semi-Markov model for a multigrade population with Poisson recruitment. J. Appl. Prob. 17 (1980) 846-852.

Mehlmann, A. Semi-Markovian manpower models in continuous time. J. Appl. Prob. 16 (1979) 416-422.

Ross, S. Applied Probability Models with Optimization Applications. Holden-Day, San Francisco, 1970.